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Metastability of Ginzburg–Landau Model with a Conservation Law

Horng-Tzer Yau[1]

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The hydrodynamics of Ginzburg–Landau dynamics has previously been proved to be a nonlinear diffusion equation. The diffusion coefficient is given by the second derivative of the free energy and hence nonnegative. We consider in this paper the Ginzburg–Landau dynamics with long-range interactions. In this case the diffusion coefficient is nonnegative only in the metastable region. We prove that if the initial condition is in the metastable region, then the hydrodynamics is governed by a nonlinear diffusion equation with the diffusion coefficient given by the metastable curve. Furthermore, the lifetime of the metastable state is proved to be exponentially large.

KEY WORDS: Metastability; hydrodynamical limit; Ginzburg–Landau dynamics; Kac potential; exponential lifetime.

1. INTRODUCTION

The hydrodynamic limit of Ginzburg–Landau models has been studied extensively in the recent literature. Results include, e.g., refs. 2, 3, 7, 5, 13, 1, and 18. Among them, the hydrodynamic limit was done in refs. 2, 3, 7, and 18, while refs. 5 and 1 dealt with large deviations and nonequilibrium fluctuations (in one dimension). In ref. 13, following the approach of ref. 7, the hydrodynamic limit was proved for Ginzburg–Landau models with finite-range interactions at any temperature. It asserts that the empirical field evolves in the hydrodynamic limit according to the nonlinear diffusive equation

$$\partial_t u(x, t) = \Delta_x [h'(u(x, t))] \tag{1.1}$$

[1] Courant Institute of Mathematical Sciences, New York University, New York, New York 10012. e-mail: yau@math1.nyu.edu.

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where $h(z)$ is the free energy of the Ginzburg–Landau model with specific magnetization (field) z. It is well known that h is always convex, but the diffusion coefficient h'' is zero in phase transition regions. Hence Eq. (1.1) becomes degenerate when there is a phase transition. In this paper, we are interested in Ginzburg–Landau models with long-range interactions, or more precisely, with the Kac potential. The Hamiltonian $H = H_0 + H_L$ is given by

$$H_0 = \sum_{j=1}^{N} V(\phi_j) \tag{1.2}$$

$$H_L = \tfrac{1}{2} N^{-a} \sum_{i=1}^{N} \sum_{j=1}^{N} J((i-j)/N^a)\phi_i \phi_j \tag{1.3}$$

where $0 < a < 1$ and J is a nonnegative function with compact support in $\{|x| < 1/4\}$ and $\int J\,dx = 1$. We shall assume the periodic boundary condition so that we are in a finite volume. The dynamics is the usual dynamics [see (2.5)], which conserves the total magnetization. Recall the definitions of pressure p and free energy h with respect to H_0,

$$p(\lambda) = \log \int \exp[\lambda\phi - V(\phi)]\,d\phi \tag{1.4}$$

$$h(x) = \sup_{\lambda} [\lambda x - p(\lambda)] \tag{1.5}$$

Define a new function $\tilde{h}$ by

$$\tilde{h}(\phi) = h(\phi) - \tfrac{1}{2}\phi^2 \tag{1.6}$$

Our main questions are: Does (1.1) hold in this case and, if (1.1) holds, what is the diffusion coefficient?

There are two natural choices for the free energy of H_L. One is the function $\tilde{h}$ defined in (1.6); the other is the convex hull $\hat{h}$ of $\tilde{h}$. Clearly, the second choice means that the result of ref. 13 holds independent of the range of the interactions. For the first choice, one should be careful because the diffusion coefficient $\tilde{h}''$ can be *negative*! Let us assume for simplicity that $\tilde{h}$ is a double-well potential and define

$$m_* = \min\{\phi \mid (\tilde{h})'(\phi) = 0\} \tag{1.7}$$

$$\underline{m} = \sup\{\phi \mid (\tilde{h})''(\phi) > 0\} \tag{1.8}$$

Thus the first choice makes sense only when initial data are smaller than $\underline{m}$.

Our main result is that the first choice holds for any $0<a<1$ [with a defined in (1.3)] provided that, roughly speaking, the initial conditions are bounded above by $\underline{m}$. Unfortunately, we cannot prove that the second choice holds for some initial data, though it is generally believed to be so. Let us describe our results in more detail. First of all, the condition that the initial data are bounded by $\underline{m}$ has to be clarified. Since the interaction has range N^a, we shall require that any local average of the field ϕ in a region of size N^{ca} with some $1>c>0$ is bounded above by $\underline{m}$. This is important because the relevant scale is N^a and local averages of size N^a should be bounded by $\underline{m}$ for the first choice to make sense. In the theorem stated in Section 2, $c=1/40$. But the method we use can be extended to any $c<1$.

The second important point is the lifetime that (1.1) holds. The usual hydrodynamic limit concerns the time scale $t \sim N^2$. If on the other hand one fixes N and lets $t \to \infty$, one obtains the invariant measure $d\mu = \exp(-H)$. Let us assume that the total specific magnetization $\bar{\phi} \equiv N^{-1} \sum_j \phi_j = m$ lies between m_* and $\underline{m}$, namely, $m_* < m < \underline{m}$. Then the invariant measure is $d\mu_m = d\mu\ \delta(\bar{\phi}=m)$. The state $d\mu_m$ describes a mixture of pure states with magnetization m_* and $m^* = \max\{\phi \mid (\tilde{h})'(\phi)=0\}$. Certainly $m^* > \underline{m}$ and the condition that local averages of ϕ are bounded by $\underline{m}$ can no longer hold. Therefore, one does not expect (1.1) to hold forever. A reasonable conjecture is that the lifetime is $\exp(\text{const} \cdot N^a)$. Again we cannot prove such a strong result, but we are able to prove the lifetime is exponentially large.

What we have described above is usually referred to as Lebowitz–Penrose[9] theory. Our results can be summarized as proving that metastability occurs as long as the interactions have range N^a with any $a>0$ and, furthermore, the lifetime of a metastable state is exponentially large. Similar results for the Kawasaki dynamics (for the Ising model with Kac potential) were obtained in refs. 8 and 15. Their results are stronger in the sense that they dealt with the infinite-volume problem. They did not, however, establish the exponential lifetime and they were restricted to the case when the parameter a in (1.3) is close to one.

Both results of refs. 8 and 15 and this paper deal with conservative dynamics. If one is interested in the nonconservative dynamics, there are extensive results on metastability for Ising models with short-range potential or Kac potential. We shall not discuss these results, as they are not directly related to this paper. We refer to, e.g., refs. 14, 16, and 6 for some recent results.

This paper is organized as follows. In Section 2 we state our main results. We then prove large deviations for short-range models in infinite volume in Sections 3 and 4. This extends results of ref. 5 in a certain sense and uses ideas from ref. 1. In Section 5 we extend these results to long-range models and prove our main results. Finally, in Section 6 we review and extend Fritz's argument[3] to control entropy productions of finite-volume systems which was used in Sections 3–5. In the Appendix, we collect some results about Eq. (1.1) and its generalizations which we need in the text.

2. STATEMENT OF MAIN RESULT

Let $\phi_j \in \mathbb{R}$, $j = 0, 1, \dots, N$, satisfy the SDE

$$d\phi_j = N^2(\Delta \partial H/\partial\phi)_j \, dt + N(d\beta_{j+1} - d\beta_j) \tag{2.1}$$

where $H = H_0 + H_L$ with

$$H_0 = \sum_{j=1}^{N} V(\phi_j) \tag{2.2}$$

$$H_L = \tfrac{1}{2} N^{-a} \sum_{i=1}^{N} \sum_{j=1}^{N} J((i-j)/N^a)\, \phi_i \phi_j \tag{2.3}$$

We shall assume the periodic boundary condition, namely $\phi_N = \phi_0$, etc. Here $0 < a < 1$ and J is a nonnegative function with support in $\{|x| < 1/4\}$ and $\int J = 1$. The Laplacian $(\Delta A)_x$ is defined as

$$(\Delta A)_j = A(j+1) - 2A(j) + A(j-1) \tag{2.4}$$

Alternatively one can describe the dynamics (2.1) by its generator. Let L be the symmetric generator characterized by

$$\begin{aligned} -\int fLf \, d\mu &= N^2 D(f) = N^2 \sum_j D_{j,j+1}(f) \\ D_j = D_{j,j+1}(f) &= \int \left(\frac{\partial f}{\partial \phi_j} - \frac{\partial f}{\partial \phi_{j+1}} \right)^2 d\mu \end{aligned} \tag{2.5}$$

where $d\mu$ is the Gibbs measure (with periodic boundary condition)

$$d\mu = e^{-H}/Z \tag{2.6}$$

Denote by $f_t(\phi)$ the density at time t relative to the measure $d\mu$. Then one can characterize (2.1) by

$$\partial_t f_t = L f_t \tag{2.7}$$

We shall assume in this paper that V is of the following type:

$$V(\phi) = \frac{1+m^2}{2} \phi^2 + \zeta(\phi) \tag{2.8}$$

with $m > 0$ and ζ a bounded function with

$$|\zeta|_\infty + |\zeta'|_\infty \leqslant \alpha_0 \tag{2.9}$$

for some constant α_0.

Let ω be a nonnegative function with support in $\{|x| < 1/4\}$ and $\int \omega = 1$. Define ω_δ by

$$\omega_\delta(x) = N^{1-\delta}\omega(N^{1-\delta}x) \tag{2.10}$$

Let $\nu_\phi^{(N)}$ be the empirical measure associated to ϕ, namely

$$\nu_\phi^{(N)} = N^{-1} \sum_{j=1}^{N} \phi_j \, \delta(j/N - z) \, dz \tag{2.11}$$

In other words, for any test function J,

$$\langle J, \nu_\phi^{(N)} \rangle = N^{-1} \sum_{j=1}^{N} J(j/N)\phi_j \tag{2.12}$$

Define $\omega * \phi$ by

$$(\omega * \phi)(x) = (\omega * \nu_\phi^{(N)})(x) \tag{2.13}$$

From now on we shall identify ϕ with $\nu_\phi^{(N)}$ whenever it is convenient without further explanation. Also, we shall drop the index N.

Strictly speaking, $\omega * \phi$ (or $\omega * \partial H/\partial\phi$) as defined in (2.13) is a measure. It is convenient to interpret $\omega * \phi$ as a density, namely

$$(\omega * \phi)(x) = N^{-1} \sum_j \omega(x - j/N)\phi_j \tag{2.13'}$$

It is also convenient to define

$$\phi(x) = \phi_{[Nx]} \tag{2.14}$$

where $[a] = \sup\{n \in \mathbb{Z}, n < a\}$. Note that with our convention $(\omega_\delta * \phi)(x)$ is an average of ϕ_j with $|j - Nx| < N^\delta$.

Recall the definition of pressure p and free energy h,

$$\begin{aligned} p(\lambda) &= \log \int \exp[\lambda\phi - V(\phi)] \, d\phi \\ h(x) &= \sup_\lambda [\lambda x - p(\lambda)] \end{aligned} \tag{2.15}$$

Clearly, these quantities are defined with respect to the free Hamiltonian H_0. Denote by the function

$$\tilde{h}(\phi) = h(\phi) - \tfrac{1}{2}\phi^2 \tag{2.16}$$

For simplicity, we assume that $\tilde{h}$ is a double-well potential. Let $\underline{m}$ be the point

$$\underline{m} = \sup\{\phi \mid (\tilde{h})''(\phi) > 0\} \tag{2.17}$$

Theorem 1. Suppose the initial data ψ for the SDE (2.1) satisfy:

(i) $\sup_x (\omega_\delta * \psi^2)(x) < C_1$ for some constant C_1 and for all $\delta \geqslant a/40$ (2.18)

(ii) There is a continuous function $v(x)$ such that $v < \underline{m} - \varepsilon_1$ and

$$\sup_x |v(x) - (\omega_\delta * \psi)(x)| < N^{-\sigma\delta} \tag{2.19}$$

for some $\sigma > 0$ and all $\delta \geqslant a/40$. Let u be the solution of the nonlinear diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= (\tilde{h}(u))_{xx} \\ u(t=0, x) &= v(x) \end{aligned} \tag{2.20}$$

Then there is a constant $\varepsilon > 0$ so that for any constant $b > 0$ and any δ with $a/40 < \delta < 1$

$$P^{\psi}\{\sup_x \sup_{0 \leqslant t < \exp(N^{\varepsilon/4})} |(\omega_\delta * \phi)(x, t) - u(x, t)| > b\} < \exp(-N^{\varepsilon}) \tag{2.21}$$

Theorem 1 is the main result of this paper. Our main tool for proving Theorem 1 is the H_{-1} norm method of ref. 1. It will be extended to give large-deviation bounds for short-range models in Sections 3 and 4. We shall then extend these results to long-range models and prove Theorem 1 in Section 5. For the rest of this section, we introduce some notations and a few preliminary comments.

Let us modify the Hamiltonian H by $\hat{H}$ in the following way:

$$\begin{aligned} \hat{H} &= H + H' = H_S + H_L \\ H' &= N^{\tau} \sum_{\sigma} g(\bar{\phi}_\sigma) \end{aligned} \tag{2.22}$$

Here σ indexes the *disjoint* box of size N^τ [$\tau = a/20$ with a defined in (2.3)] and $\bar{\phi}_\sigma$ is the average of ϕ in the box σ. The function g is chosen so that

$$\hat{h}'' = h'' + g'' - 1 > c_2 > 0 \tag{2.23}$$

$$g(x)'' \geqslant 0 \quad \text{for all } x, \qquad g(a) = 0 \quad \text{for} \quad x < \underline{m} - \varepsilon_1/2 \tag{2.24}$$

$$|g|_\infty + |g'|_\infty \leqslant \alpha_1 \tag{2.25}$$

for some constant c_2 and α_1.

With the Hamiltonian $\hat{H}$, we can define the dynamics reversible with respect to $\hat{H}$ by replacing H in (2.1) by $\hat{H}$. Let A denote the event in (2.21) with $b \leqslant \varepsilon_1/4$. Clearly, when restricted to the event A^c, the dynamics (2.1) is identical to the new dynamics with H replaced by $\hat{H}$, since $H' = 0$ on A^c. Hence, if we can prove (2.21) with respect to the dynamics generated by $\hat{H}$, we conclude (2.20) for H as well. The Hamiltonian $\hat{H}$ has the advantage that the diffusion coefficient $\hat{h}''$ is strictly positive in (2.23), which is not satisfied by $\tilde{h}''$. For the rest of this paper we shall be concerned only with the modified Hamiltonian $\hat{H}$ and prove Theorem 1 for $\hat{H}$ only.

The following notations will be used throughout this paper.

Let K_α be the kernel defined by

$$K_\alpha(x, y) = \frac{\alpha^{-1}}{2} \exp[-\alpha |x - y| - \theta(\alpha x) - \theta(\alpha y)] \tag{2.26}$$

Here α is a small constant and θ is a smooth function with $\theta(0) = 0$ and $\theta(x) = |x|$ for x large. Define the norms

$$\|u\|_0^2 = \int u(x)^2 \, e^{-2\theta(\alpha x)} \, dx \tag{2.27}$$

$$\|u\|_{-1}^2 = \iint u(x) \, u(y) \, K_\alpha(x, y) \, dx \, dy \tag{2.28}$$

Denote the corresponding inner product by $\langle \cdot \rangle_0$, $\langle \cdot \rangle_{-1}$. For convenience of later reference, we list here some properties of K:

$$\partial_y^2 K(x, y) = -\delta(x - y) \exp[-2\theta(\alpha x)] + \alpha^2 [1 + U(x, y)] \, K(x, y) \tag{2.29}$$

$$U(x, y) = \theta'(y) \operatorname{sign}(x - y) + \theta'(y)^2 - \theta''(y) \tag{2.30}$$

$$\begin{aligned}(N^2 \Delta_y K)(x, y) = {} & -\delta(x - y) \exp[-2\theta(\alpha x)] \\ & + \alpha^2 [1 + U(x, y) + O(1/N)] \, K(x, y) \\ & + s(x, y) \exp[-\theta(\alpha x) - \theta(\alpha y)]\end{aligned} \tag{2.31}$$

Here s is defined by

$$s(x, y) = -\alpha N [1 - N|x - y| + O(1/N)] \, 1(|y - x| < N^{-1}) \tag{2.32}$$

3. KEY ESTIMATE

Our goal in this section is to prove a large-deviation bound for short-range Ginzburg–Landau models in infinite volume. For the rest of this section, we shall concern ourselves only with (2.1) with $H=H_0$ and V given by (2.8) and (2.9). Our results can be extended easily to models with short-range interactions and the function V need not be of the special form given by (2.8) and (2.9). We shall content ourselves with the simplest case in order to focus on the main ideas. In the next section, these bounds will be extended to long-range models.

Let us first review the role of the H_{-1} norm in proving the uniqueness of the nonlinear diffusion equation. Let J be nonnegative function with $\int J(x)\,dx=1$. Let b be a smooth function satisfying for some $\varepsilon>0$,

$$\varepsilon^{-1}>\inf_{x\in\mathbb{R}} b'(x)>1+\varepsilon \tag{3.1}$$

Let u and v be two solutions to the equation

$$\partial_t w=[b(w)-J*w]_{xx}, \qquad w(t=0)=w_0 \tag{3.2}$$

The following lemma shows that the H_{-1} norm of $u-v$ is a contraction. More precisely

Lemma 3.1. There exists a constant C such that for α in (2.26) small enough,

$$\frac{d}{dt}\|u-v\|_{-1}^2\leqslant -C\|u-v\|_0^2 \tag{3.3}$$

Proof. By definition of $\|\cdot\|_{-1}$,

$$\frac{d}{dt}\|u-v\|_{-1}^2=\iint (u-v)(x)\partial_y^2K_\alpha(x,y)$$
$$\times[b(u(y))-b(v(y))-J*(u-v)(y)]\,dx\,dy \tag{3.4}$$

By (2.29), $\partial_y^2K_\alpha(x,y)$ has two main terms. Let us denote the corresponding contributions by Ω_i, $i=1,2$. By definition Ω_1 is equal to

$$\Omega_1=-\int (u-v)(x)[b(u(x))-b(v(x))]e^{-2\theta(\alpha x)}\,dx$$
$$+\int (u-v)(x)[J*(u-v)](x)e^{-2\theta(\alpha x)}\,dx$$
$$\equiv W_1+W_2 \tag{3.5}$$

From assumption (3.1), we can bound W_1 by

$$W_1 \leqslant -(1+\varepsilon)\|u-v\|_0^2 \tag{3.6}$$

By the Schwartz inequality, W_2 is bounded by

$$W_2 \leqslant \tfrac{1}{2}\|u-v\|_0^2 + \tfrac{1}{2}\|J * (u-v)\|_0^2 \tag{3.7}$$

By definition of θ, we have $\|\theta'\|_\infty \leqslant$ const and hence the last term in (3.7) is bounded by $[1+\text{const}(\alpha)]\|u-v\|_0^2/2$ with $\text{const}(\alpha) \to 0$ as $\alpha \to 0$. Combining (3.6) and (3.7), we have proved that

$$\Omega_1 < -\varepsilon/2\,\|u-v\|_0^2 \tag{3.8}$$

provided that α is small enough.

We now bound Ω_2. By definition

$$\begin{aligned}\Omega_2 = {} & \alpha^2\langle u-v, b(u)-b(v)-J*(u-v)\rangle_{-1} \\ & + \alpha^2 \iint (u-v)(x)\, U(x,y)\, K_\alpha(x,y) \\ & \times [b(u(y))-b(v(y))-J*(u-v)(y)]\, dx\, dy \end{aligned} \tag{3.9}$$

It is easy to bound Ω_2 by

$$\Omega_2 \leqslant \alpha^2 \,\text{const}(\varepsilon)\,\|u-v\|_0^2 \tag{3.10}$$

Here we have used (3.1). By choosing α small enough, we conclude (3.3) from (3.8) and (3.10). ∎

Lemma 3.1 can be extended to the SDE (2.1) in infinite volume in the following sense.

Lemma 3.2. Suppose $n(x, t)$ is the solution to the equation [with h defined in (2.15)]

$$\partial_t n(x,t) = h'(n(x,t))_{xx} \equiv \partial_x \partial_x h'(n(x,t)), \qquad n(x, t=0) = n_0(x) \tag{3.11}$$

where n_0 is the initial condition satisfying

$$\|n_0\|_0^2 \leqslant c_0 \tag{3.12}$$

Assume that h' satisfies that for some $\eta > 0$ the bound

$$\infty > \eta^{-1} > h' > \eta > 0 \tag{3.13}$$

Let $\omega = \omega_\delta$ be the characteristic function (normalized)

$$\omega(x) = \tfrac{1}{2} N^{1-\delta}\, 1(|x| \leqslant N^{\delta - 1}) \tag{3.14}$$

where δ is a positive constant less than 1. Define G_1 and G_2 by

$$G_i = \int g_i(x) e^{-2\theta(\alpha x)}\, dx, \qquad i = 1, 2$$

$$g_1(x) = 1 - [\omega * (\phi\, \partial H/\partial\phi)](x) + (\omega * \phi)(x)(\omega * \partial H/\partial\phi)(x) \tag{3.15}$$

$$g_2(x) = [\omega * \partial H/\partial\phi - h'(\omega * \phi)]^2(x) \tag{3.16}$$

Define Ω and dM by the equation

$$\tfrac{1}{2} d\, \|\phi - n\|^2_{-1}(t) = \Omega(t)\, dt + dM(t) \tag{3.17}$$

where dM is the Martingale part. The there is an $l(t)$ depending on C_0, α in (2.26), and ζ in (2.8) such that for α small enough $(n_x \equiv \partial_x n)$

$$\Omega(t) \leqslant -C_1 \|\omega * (\phi - n)\|_0^2(t) + C_2(G_1 + G_2)(t) + C_3(t)\, N^{\delta-1}[1 + \|n_x\|_0^2](x) \tag{3.18a}$$

if $\|\phi\|_0(t) < l(t)$ and

$$\Omega(t) \leqslant -C_4 \|\phi\|_0^2 \tag{3.18b}$$

if $\|\phi\|_0(t) \geqslant l(t)$. Here C_i are positive constants. Furthermore, the quadratic variation of dM satisfies the bound

$$(dM)^2(t) \leqslant C_5 N^{-1} \|\phi - n\|_0^2(t) \tag{3.19}$$

The main estimate of Lemma 3.2 is (3.18a). The bound (3.18b) simply provides a cutoff for the large field. Compared with (3.4), there are two more correction terms in (3.18a). The last term of (3.18a) is negligible as $N \to \infty$. The term with G_1 and G_2 is order one, but it involves only fluctuations. More precisely, G_1 and G_2 are negligible if the density f_t is a local Gibbs state. Following ref. 7 or ref. 1, this can be done via a Dirichlet form estimate (see Lemmas 3.4–3.6). Finally we remark that the constant 1 in (3.15) comes from the quadratic term in the Ito calculus and does not appear in the setting of differential equations in Lemma 3.1.

Proof. We shall omit the index α on K.

Step 1. By Ito's formula and integration by parts, $\Omega = \Omega_1 + \Omega_2 + \Omega_3$ with

$$\Omega_1 = \iint (\phi - n)(x)(N^2 \Delta_y K(x, y))(\partial H/\partial\phi)(y)\, dx\, dy \tag{3.20}$$

$$\Omega_2 = -\iint (\phi - n)(x)\, \partial_y^2 K(x, y)\, h'(n(y))\, dx\, dy \tag{3.21}$$

$$\Omega_3 = \frac{1}{2} \sum_i \left[K\left(\frac{i}{N}, \frac{i}{N}\right) - 2K\left(\frac{i}{N}, \frac{i+1}{N}\right) + K\left(\frac{i+1}{N}, \frac{i+1}{N}\right) \right] \tag{3.22}$$

Here and for the rest of this section, all summations for i and j are summing over all integers. Also, dM is equal to

$$dM = N^{-1} \sum_j \int (\phi - n)(x)(N\nabla_y K(x, j/N))\, dx\, d\beta(j/N) \tag{3.23}$$

We can compute Ω_3 by

$$\Omega_3 = N^{-1} \sum_i e^{-2\theta(\alpha i/N)} + O(N^{-1}) \tag{3.24}$$

The quadratic variation of dM can also be computed:

$$\begin{aligned}(dM)^2/dt &= N^{-2} \sum_j \left[\int (\phi(x) - n(x))(N\nabla_y K(x, j/N))\, dx \right]^2 \\ &\leqslant N^{-2} \left\{ \sum_j \int [N\nabla_y K(x, j/N)]^2 e^{2\theta(\alpha x)}\, dx \right\} \|\phi - n\|_0^2 \\ &\leqslant \text{const} \cdot N^{-1} \|\phi - n\|_0^2 \end{aligned} \tag{3.25}$$

Here we have used the Schwartz inequality in the first inequality. This proves (3.19).

Step 2. We now bound Ω_1. Let us first use (2.30) to write Ω_1 as $\Omega_1 = \Omega_4 + \Omega_5 + O(1/N)$ with

$$\Omega_4 = -\int (\phi - n)(x)(\partial H/\partial\phi)(x) \exp[-2\theta(\alpha x)]\, dx \tag{3.26}$$

$$\Omega_5 = \alpha^2 \iint (\phi - n)(x)[1 + U(x, y)] \cdot K(x, y)(\partial H/\partial\phi)(y)\, dx\, dy \tag{3.27}$$

It is not hard to see that one can replace $(\phi - n)(x)$ by $\omega * (\phi - n)(x)$ and $(\partial H/\partial\phi)(x)$ by $(\omega * \partial H/\partial\phi)(x)$ in Ω_5 with small error. More precisely, let Ω_7 denote Ω_5 with $(\phi - n)$ and $\partial H/\partial\phi$ replaced by $\omega * (\phi - n)$ and $\omega * \partial H/\partial\phi$, respectively. Then by simple computations and the Schwartz inequality one has the bound

$$\begin{aligned}|\Omega_5 - \Omega_7| &\leqslant \text{const} \cdot N^{2(\delta - 1)}[\|\phi - n\|_0^2 + \|\partial H/\partial\phi\|_0^2] \\ &\leqslant \text{const} \cdot N^{2(\delta - 1)}[\|\phi - n\|_0^2 + 1 + \|n\|_0^2]\end{aligned} \tag{3.28}$$

Here δ is the constant in (3.14) and we have used the bound $|\partial H/\partial\phi|(x) \leqslant \text{const} \cdot [|\phi - n| + |n| + 1](x)$ in the last inequality.

Next we replace $(\phi - n) \cdot (\partial H/\partial\phi)$ in Ω_4 by $\omega * (\phi - n) \cdot (\omega * \partial H/\partial\phi)$ and denote it by Ω_6. By definition of G_1 and (3.24),

$$\Omega_4 = -\Omega_3 + G_1 + \Omega_6 + \Omega_8 + \Omega_9 + O(1/N) \tag{3.29}$$

where Ω_8 and Ω_9 are defined by

$$\Omega_8 = \int \{[\omega * (\phi\, \partial H/\partial\phi)](x) - [\phi\, \partial H/\partial\phi](x)\} \exp[-2\theta(\alpha x)]\, dx \tag{3.30}$$

$$\Omega_9 = \int \{n(x)(\partial H/\partial\phi)(x) - (\omega * n)(x)(\omega * \partial H/\partial\phi)(x)\} \exp[-2\theta(\alpha x)]\, dx \tag{3.31}$$

Since ω has range $N^{\delta-1}$ and θ is smooth, Ω_8 can be bounded by

$$\Omega_8 \leqslant \text{const} \cdot N^{2\delta-2}[\|\phi\|_0^2 + 1] \tag{3.32}$$

Here we have used (2.8) and (2.9). Similarly we can bound Ω_9 by

$$\begin{aligned}\Omega_9 &\leqslant \langle |n - \omega * n|, |\partial H/\partial\phi| \rangle_0 + \left\langle |\omega * n|, \left|\partial H/\partial\phi - \omega * \frac{\partial H}{\partial\phi}\right| \right\rangle_0 \\ &\leqslant N^{1-\delta} \|\omega * n - n\|_0^2 + N^{\delta-1} \|\partial H/\partial\phi\|_0^2 + N^{\delta-1} \|\omega * n\|_0^2 \\ &\quad + N^{1-\delta} \|\partial H/\partial\phi - \omega * \partial H/\partial\phi\|_0^2 \\ &\leqslant \text{const} \cdot N^{\delta-1}[\|\phi\|_0^2 + 1 + \|n\|_0^2] + N^{1-\delta} \|\omega * n - n\|_0^2 \end{aligned} \tag{3.33}$$

By the Poincaré inequality, $\|\omega * n - n\|_0^2 \leqslant N^{2\delta-2} \|n_x\|_0^2$. Hence Ω_9 is bounded by

$$\Omega_9 \leqslant \text{const} \cdot N^{\delta-1}[1 + \|\phi\|_0^2 + \|n\|_0^2 + \|n_x\|_0^2] \tag{3.34}$$

Let us summarize what we have proved so far:

$$d\|\phi - n\|_{-1}^2 \leqslant \Omega_2 + \Omega_6 + \Omega_7 + G_1 + \text{const}(N)\, N^{\delta-1}[1 + \|n_x\|_0^2 + \|\phi\|_0^2] + dM \tag{3.35}$$

Here we have used Lemma A.1 to bound $\|n\|_0^2(t)$ by $\text{const}(t) \|n\|_0^2(0)$ and adsorbed it into the constant term.

Step 3. In step 2 we have replaced $\phi \cdot \partial H/\partial\phi$ by $(\omega * \phi) \cdot (\omega * \partial H/\partial\phi)$ and showed that the error is essentially G_1. Our goal in this step is to replace $\omega * \partial H/\partial\phi$ by $h'(\omega * \phi)$ and show that the error is essentially bounded by G_2. Let us define Ω_{10} and Ω_{11} by

$$\Omega_{10} = -\langle \omega * (\phi - n), h'(\omega * \phi)\rangle \tag{3.36}$$

$$\Omega_{11} = \alpha^2 \iint [\omega * (\phi - n)](x)[1 + U(x, y) + O(1/N)] \times K(x, y)\, h'(\omega * \phi)(y)\, dx\, dy \tag{3.37}$$

By the Schwartz inequality, the differences $\Omega_{10} - \Omega_6$ and $\Omega_{11} - \Omega_7$ can be bounded as [with α denoting the constant in (2.26)]

$$|\Omega_{10} - \Omega_6| + |\Omega_{11} - \Omega_7| \leqslant \alpha \|\omega * (\phi - n)\|_0^2 + \alpha^{-1} G_2 \tag{3.38}$$

We have thus proved that

$$d\|\phi - n\|_{-1}^2 \leqslant \{\Omega_{12} + \Omega_{13} + G_1 + \alpha^{-1} G_2 + \text{const} \cdot N^{\delta - 1} \cdot [1 + \|n_x\|_0^2 + \|\phi\|_0^2]\}\, dt + dM \tag{3.39}$$

where Ω_{12} and Ω_{13} are defined by

$$\Omega_{12} = \iint (\omega * \phi - \omega * n)(x)\, \partial_y^2 K(x, y)[h'(\omega * \phi) - h'(\omega * n)](y)\, dx\, dy \tag{3.40}$$

$$\Omega_{13} = \alpha \|\omega * (\phi - n)\|_0^2 \tag{3.41}$$

Note that Ω_{12} is similar to the right side of (3.4) with $J = 0$. Hence the same argument yields that

$$\Omega_{12} \leqslant -[\eta - \text{const} \cdot \alpha]\, \|\omega * (\phi - n)\|_0^2 \tag{3.42}$$

This proves (3.18a) provided that α is small enough.

Step 4. We now suppose $\|\phi\|_0^2(t) > l$. By definition of H, Ω_4 is bounded by

$$\begin{aligned} \Omega_4 &= -\langle \phi, \partial H/\partial\phi\rangle_0 + \langle n, \partial H/\partial\phi\rangle_0 \\ &\leqslant -\text{const} \cdot \|\phi\|_0^2 + \text{const} \cdot \langle |\phi|, 1\rangle_0 + \varepsilon^{-1} \|n\|_0^2 + \varepsilon \|\partial H/\partial\phi\|_0^2 \\ &\leqslant -\text{const} \cdot \|\phi\|_0^2 + \varepsilon^{-1} \|n\|_0^2 + \text{const} \end{aligned} \tag{3.43}$$

Here we have used the Schwartz inequality in the second inequality with ε sufficiently small. Since $\|n\|_0^2(t) \leqslant \text{const}(t)\, \|n\|_0^2(0)$ by Lemma A.1, we can choose l large enough so that

$$\Omega_4 \leqslant -\text{const} \cdot \|\phi\|_0^2 \tag{3.44}$$

provided that $\|\phi\|_0^2 > l(t)$. It is not hard to prove that Ω_5 is bounded by

$$\Omega_5 \leqslant \text{const} \cdot \alpha^2 [\|\phi\|_0^2 + 1] \tag{3.45}$$

By choosing α small enough, Ω_5 can be controlled by Ω_4. This proves (3.18b). ▮

Using the exponential Martingale, one can easily reformulate Lemma 3.2 as follows:

Lemma 3.3. Under the assumption of Lemma 3.2, we have for $\delta < 1/2$ and $\gamma < 1$

$$\begin{aligned} E^{\psi}\bigg[\exp\bigg\{\tfrac{1}{2}N^{\delta}\gamma\bigg[\|\phi - n\|_{-1}^2(t) - \|\phi - n\|_{-1}^2(0) \\ + \text{const} \cdot \int_0^t \|\phi\|_0^2 \, 1(\|\phi\|_0^2 \geqslant l(s))\, ds\bigg]\bigg\}\bigg] \\ \leqslant \text{const} \cdot E^{\psi}\bigg[\exp\bigg\{\text{const} \cdot \gamma N^{\delta} \int_0^t (G_1 + G_2)(s)\, 1(\|\phi\|_0^2(s) < l(s))\, ds\bigg\}\bigg] \end{aligned} \tag{3.46}$$

Here the const depends on t.

Proof. Let ζ_t be defined by

$$\begin{aligned} \zeta_t = \gamma N^{\delta} \|\phi - n\|_{-1}^2(t) - \gamma N^{\delta} \|\phi - n\|_{-1}^2(0) \\ - \gamma N^{\delta} \int_0^t \Omega(s)\, ds - \frac{\gamma^2}{2} N^{2\delta} \int_0^t (dM)^2(s) \end{aligned} \tag{3.47}$$

where Ω is defined by (3.17). Then $\exp[\zeta_t]$ is a Martingale and hence

$$E[\exp\{\xi_t\}] = 1 \tag{3.48}$$

By Lemma 3.2, ζ_t is bounded by

$$\begin{aligned} \zeta_t \geqslant \gamma N^{\delta} \|\phi - n\|_{-1}^2(t) - \gamma N^{\delta} \|\phi - n\|_{-1}^2(0) \\ - \text{const} \cdot \gamma N^{\delta} \int_0^t (G_1 + G_2)(s)\, 1(\|\phi\|_0(s) < l(s))\, ds \\ + \text{const} \cdot \gamma N^{\delta} \int_0^t \|\phi\|_0^2 \, 1(\|\phi\|_0(s) \geqslant l(s))\, ds \\ + \text{const} \cdot \gamma N^{\delta - 1} \int_0^t [1 + \|n_x\|_0^2](s)\, ds \\ + \text{const} \cdot \gamma^2 N^{2\delta - 1} \int_0^t [\|\phi\|_0^2 + \|n\|_0^2](s)\, ds \end{aligned} \tag{3.49}$$

Here we have used (3.18a) and (3.18b) for Ω and (3.19) for $(dM)^2$. By Lemma A.1, $\int_0^t \|n_x\|_0^2(s)\,ds$ is bounded. Together with the fact that $\|n\|_0^2$ is bounded, the last three terms of (3.49) are bounded by

$$\text{const}\cdot(\gamma N^{\delta}-\gamma^2 N^{2\delta-1})\int_0^t \|\phi\|_0^2\, 1(\|\phi\|_0^2(s)\geqslant l(s))\,ds$$
$$+\text{const}\cdot(\gamma^2 N^{2\delta-1}+\gamma N^{2\delta-1}) \tag{3.50}$$

Here the constants depend on l and t. Recall that by assmption $\delta\leqslant 1/2$. Hence for N large we have

$$\zeta_t\geqslant \gamma N^{\delta}\|\phi-n\|_{-1}^2(t)-\gamma N^{\delta}\|\phi-n\|_{-1}^2(0)$$
$$-\text{const}\cdot\gamma N^{\delta}\int_0^t (G_1+G_2)(s)\,1(\|\phi\|_0^2(s)\leqslant l(s))\,ds$$
$$+\text{const}\cdot\gamma N^{\delta}\int_0^t \|\phi\|_0^2(s)\,1(\|\phi\|_0^2\geqslant l(s))\,ds+\text{const}(\gamma) \tag{3.51}$$

Lemma 3.3 follows from (3.48), (3.51), and the Hölder inequality. ∎

Remark. The "constant" l actually depends on t. But since all our arguments in this paper will be carried out for t fixed, one can always replace $l(s)$ by $\sup_{0<s<t} l(s)$ or $\inf_{0<s<t} l(s)$. From now on, we will treat l as a constant independent of time.

The right side of (3.46) can be bounded by the Portenko lemma.[17] We state it as follows.

Lemma 3.4. For any nonnegative function G let ρ be defined by

$$\rho=\sup_{0\leqslant t'\leqslant t}\ \sup_{\|\xi\|_0<l} E^{\xi}\left[\int_0^{t'} G(s)\,1(\|\phi\|_0(s)<l)\,ds\right] \tag{3.52}$$

Then

$$E^{\psi}\left[\exp\left\{\int_0^t G(s)\,1(\|\phi\|_0(s)<l)\right\}\right]\leqslant(1-\rho)^{-1} \tag{3.53}$$

Remark. Lemma 3.4 is slightly different from the Portenko lemma in ref. 17. The characteristic function $1(\|\phi\|_0(s)<l)$ makes it possible to include the condition $\|\xi\|_0<l$ in the definition of ρ.

Finally we have to bound ρ. Let f_t satisfy (2.7) with initial condition a δ-function of a configuration ξ. By the theorem of ref. 10, the entropy of f_t, $s(f_t)$ in (6.11), is bounded for $t=N^{-2}$ provided that $\|\xi\|_0^2\leqslant$const. Also, it is easy to check that $E^{\xi}[\|\phi\|_0^2(\tau)] \leqslant \text{const}$ for $0 \leqslant \tau \leqslant N^{-2}$. Hence for G satisfying $0 \leqslant G \leqslant \text{const} \cdot \gamma N^{\delta} \|\phi\|_0^2$ we have that

$$\rho \leqslant \text{const} \cdot \gamma N^{\delta - 2} + u \tag{3.54}$$

$$u = \sup_{s(f) \leqslant \text{const}} \sup_{0 \leqslant t' \leqslant t} E^{f}\left[\int_0^{t'} G(s)\, 1(\|\phi\|_0(s) < l(s))\, ds\right] \tag{3.55}$$

In the application we have in mind, $\gamma N^{\delta - 2} \ll 1$. So we need to decide, say, when $u < 1/2$. We follow the approach of refs. 7, 3, and 1 and estimate u by the Dirichlet form. More precisely, let $f_{t,j}$ be the marginal of f_t on $[-j, j]$. Then by Theorem 6.1 the Dirichlet form of f_t is bounded in the sense that

$$\sum_j \sum_{i=1}^{j-2} \int_0^t E^{\mu}[(\partial_i - \partial_{i+1}) f_{\sigma,j}]^2 f_{\sigma,j}^{-1}\, d\sigma \exp[-\theta(\alpha j/N)] \leqslant q s(f_0) \tag{3.56}$$

for some constant q. The constant q is of order 1 here and is not important. In Section 4, it will be of order N^a for some $a > 0$.

Let us assume now G is either $\gamma N^{\delta} |G_1|$ or $\gamma N^{\delta} G_2$, which is our main interest. Consider the eigenvalue problem $(i = 1, 2)$

$$\varepsilon_i(b, \gamma, N, \delta, y) = \sup_{h \geqslant 0,\, \int h\, d\mu = 1} \left\{\gamma N^{\delta} \int |g_i|(x = 0) h\, d\mu_y \right.$$
$$\left. - b N^{2-\delta} q^{-1} \sum_{-N^{\delta} \leqslant j < N^{\delta}} \int [(\partial_j - \partial_{j+1}) h]^2 h^{-1}\, d\mu_j \right\} \tag{3.57}$$

Here $d\mu_y$ is the canonical Gibbs state defined by

$$d\mu_y = \left\{\prod_{|j| \leqslant N^{\delta}} \exp[-V(\phi_j)]\, d\phi_j\right\} \delta\left(\frac{1}{1 + 2N^{\delta}} \sum_{|j| \leqslant N^{\delta}} \phi_j - y\right) \Big/ \text{normalization} \tag{3.58}$$

Let $\varepsilon_i(b, \gamma, N, \delta)$ be the sup of $\varepsilon_i(b, \gamma, N, \delta, y)$ over all possible y, namely

$$\varepsilon_i(b, \gamma, N, \delta) = \sup_y \varepsilon_i(b, \gamma, N, \delta, y) \tag{3.59}$$

By (3.56), one has $(i = 1, 2)$

$$\left|\gamma N^{\delta} \iint_0^t G_i(s) f(s)\, ds\, d\mu\right| \leqslant \varepsilon_i(b, \gamma, N, \delta) + s(f_0)/b \tag{3.60}$$

Note that (3.60) follows by averaging (3.57) with respect to x weighted by $\exp[-2\theta(\alpha x)$. Strictly speaking, (3.57) is independent of x. Hence a translation of x on the right side of (3.57) is understood when averaging w.r.t. x is taken.

We now summarize what we have proved as the following result.

Lemma 3.5. The constant u_i (with $G=\gamma N^\delta |G_i|$) is bounded by

$$u_i \leqslant \varepsilon_i(b, \gamma, N, \delta) + s(f_0)/b, \qquad i=1,2 \tag{3.61}$$

Finally, we have to decide when ε_i is small.

Lemma 3.6. Suppose that q is bounded by

$$q \leqslant N^{2-e} \tag{3.62}$$

for some $e>0$. Then for

$$\delta \leqslant e/5, \qquad \gamma \leqslant N^{-3\delta/4} \tag{3.63}$$

we have that

$$\varepsilon_i(b, \gamma, N, \delta) \to 0 \qquad \text{as} \quad N \to \infty \tag{3.64}$$

for any constant b.

Proof. Let $\delta = e/5$ ($\delta < e/5$ is similar). Hence

$$\varepsilon_i(b, \gamma, N, \delta, y) = bN^{3\delta} \sup_h \left\{ \gamma b^{-1} N^{-2\delta} \int |g_i| (x=0) h \, d\mu_y - N^{2\delta} \sum_{-N^\delta \leqslant j < N^\delta} \int [(\partial_j - \partial_{j+1}) h]^2 h^{-1} \, d\mu_y \right\} \tag{3.65}$$

By the logarithmic Sobolev inequality for the product measure,[11,12] we can replace the Dirichlet form by the relative entropy to have an upper bound. Hence

$$\varepsilon_i(b, \gamma, N, \delta, y) \leqslant bN^{3\delta} \sup_h \left\{ \gamma b^{-1} N^{-2\delta} \int |g_i| (x=0) h \, d\mu_y - \text{const} \cdot S(h/\mu_y) \right\} \tag{3.66}$$

where the entropy S is defined in (6.8). By the entropy inequality (6.7), ε_i is bounded by

$$\varepsilon_i(b, \gamma, N, \delta, y) \leqslant bN^{3\delta} \log \int \exp\{\text{const} \cdot \gamma b^{-1} N^{-2\delta} |g_i| (x=0)\} \, d\mu_y \tag{3.67}$$

This integration was studied in ref. 1 and it was proved that

$$\varepsilon_i(b, \gamma, N, \delta, Y) \leqslant \text{const} \cdot bN^{3\delta} [\gamma b^{-1} N^{-2\delta - \delta/3} + (\gamma b^{-1} N^{-2\delta})^2] \tag{3.68}$$

Granting this bound, we have that $\varepsilon_i \to 0$ is $N \to \infty$ provided that γ satisfies (3.63). This proves (3.64).

We now return to sketch the proof of (3.68), which is elementary but somewhat lengthy.[1] Let us first assume that $g_i(x=0)$ is bounded. Then by expanding the exponential up to second order we have that the expectation is bounded by (up to the second order)

$$1 + \text{const} \cdot \gamma b^{-1} N^{-2\delta} \int |g_i| (x=0)\, d\mu_y + \text{const} \cdot (\gamma b^{-1} N^{-2\delta})^2 \quad (3.69)$$

It suffices to show that

$$\int |g_i| (x=0)\, d\mu_y \leqslant \text{const} \cdot N^{-\delta/3} \quad (3.70)$$

Let $d\mu_\lambda$ be the product measure with the chemical potential λ instead of the constraint y. Certainly, λ is chosen so that the expectation of $(1+2N^\delta)^{-1} \sum \phi_j$ is equal to y.

A strong form of the equivalence of ensembles states that

$$\left| \int g\, d\mu_y - \int g\, d\mu_\lambda \right| \leqslant \text{const} \cdot \|g\|_\infty N^{-\delta/2} \quad (3.71)$$

Since $g_i(x=0)$ satisfies that

$$\int |g_i| (x=0)\, d\mu_\lambda \leqslant \text{const} \cdot N^{-\delta/2} \quad (3.72)$$

we have thus proved (3.70). To complete this argument, one has to perform cutoff for large g_i and prove the equivalence of the ensemble bound (3.71). The equivalence of the ensemble (3.71) was proved in ref. 1, while the cutoff for g_i is very straightforward since g_i is quadratic when ϕ becomes large. We omit the details. ∎

Combining Lemmas 3.3–3.6, we have the following two corollaries. Corollary 3.7 is a statement of large deviation.

Corollary 3.7. Suppose that (3.55) holds for $g \leqslant N^{2-e}$ for some $e>0$. Let $\delta = e/5$. Then for ψ satisfying $\|\psi\|_0^2 \leqslant \text{const}$ one has

$$E^\psi[\exp\{\tfrac{1}{2} N^{\delta/4} \|\phi - n\|_{-1}^2(t)\}] \leqslant \text{const}(t) \quad (3.73)$$

Here n is a solution to (3.11) with $n_0 = \psi$.

Corollary 3.8. Suppose that ψ satisfies

$$\|\psi\|_{-1}^2 \leqslant \text{const}$$

Then for l large enough

$$E^{\psi}\left[\exp\left\{\tfrac{1}{2}N\int_0^t \|\phi\|_0^2\, 1(\|\phi\|_0 > l)\, ds\right\}\right] \leqslant \exp(\text{const}\cdot N) \tag{3.74}$$

4. LARGE-DEVIATION BOUND FOR SHORT-RANGE MODEL

In this section we shall prove Theorem 4.1, which is similar to Theorem 1, but in the context of short-range models. Our setting will be as in Section 3, except we are now *in finite volume*. This assumption is only used in Lemma 4.2 to provide some simple cutoff without too much work. Roughly speaking, our main result can be described as follows. In ref. 5 the large deviation for Ginzburg–Landau models was proved. Their result, however, covers only "small macroscopic regions," namely, for the average of the field ϕ_j with $|j-i| < N\delta$ and δ small. If one is interested in a region, say, $|j-i| < N^{\sigma}$ for some $\sigma > 0$, then no conclusion can be drawn from ref. 5. The following Theorem 4.1 provides such a bound in a rather strong sense. Throughout this section, our dynamics is governed by (2.1) with $H = H_0$. Recall that ω is a smooth function with compact support and ω_δ of (2.10) has support in $\{x \mid |x| \leqslant N^{\delta-1}\}$. We now state the main result of this section. Note that $\omega_{1-\varepsilon}(x)$ in the theorem has support in $|x| \leqslant N^{-\varepsilon}$.

Theorem 4.1. Suppose that the initial condition ψ satisfies that:

(A) There is a smooth function $u_0(x)$ such that

$$\sup_x |u_0(x) - (\omega_{1-\varepsilon} * \psi)(x)| < N^{-\sigma(1-\varepsilon)} \tag{4.1}$$

for some $0 < \varepsilon < 1$ and $0 < \sigma < 1$.

(B)

$$\|\psi\|_0 \leqslant \text{const} \tag{4.2}$$

Let $u(x, t)$ solve (3.11). Then there is a $\gamma > 0$ such that

$$E^{\psi}\Big[\sup_{0<t<\exp(N^{\gamma/4})} \sup_x |u(x,t) - (\omega_{1-\varepsilon} * \psi)(x,t)| > N^{-\gamma}\Big] < \exp(-N^{\gamma}) \tag{4.3}$$

We start our proof with the following Lemma 4.2, which provides basic cutoff for large ϕ. Lemma 4.2 is the only place we use the finite-volume assumption.

Lemma 4.2. Suppose the initial condition satisfies (4.2). Then for $C_1 > 0$ large enough there is a $C_2 > 0$ such that

$$E^{\psi}\{\sup_{0<t<\exp(\text{const}\cdot N)} \|\phi\|_0^2(t) > C_2\} < \exp(-C_1 N) \tag{4.4}$$

Proof. First of all we claim that (4.4) holds if the dynamics starts from the equilibrium. This has been proved in the proof of Lemma 6.1 of ref. 7. To extend (4.4) to nonequilibrium, we note that (4.4) holds if the initial density f satisfies $\int f^2\, d\mu \leqslant \exp(\text{const}\cdot N)$. But a simple extension of the theorem of ref. 10 shows that the density of the system f_t satisfies that $\int f_t^2\, d\mu \leqslant \exp(\text{const}\cdot N)$ for $t = N^{-2}$. It remains to prove (4.4) for $0 < t \leqslant N^{-2}$. But this is just an elementary application of Ito's calculus. ∎

Lemma 4.3. Suppose that the initial condition ψ satisfies $\|\psi\|_{-1}^2 < C$ for some constant C. Let τ_l be the first hitting time of

$$A_l = \{\|\phi\|_0^2 < l\} \tag{4.5}$$

Then for l large enough

$$E^{\psi}[\tau_l > 1/4] < \exp(-\text{const}\cdot N) \tag{4.6}$$

Proof. One simply applies (3.74) and the Chebyshev inequality. ∎

In the following lemma, we shall extract results from Corollary 3.7. It states that the averages of local field at two different scales should be almost the same with very high probability.

Lemma 4.4. Suppose that the initial condition satisfies that $\|\psi\|_{-1}^2 \leqslant \text{const}$. Then

$$\sup_A E^{\psi}\{|(\omega_{1-\delta/12} * \phi)(x, t) - (\omega_{1-\delta/24} * \phi)(y, x)| > N^{-\delta/24}\}$$
$$< \exp(-\text{const}\cdot N^{\delta/12}) \tag{4.7}$$

Here δ is chosen as in Corollary 3.7 and A is defined by

$$A = \{|x-y| + |t-s| < N^{-\delta/24}; \tfrac{1}{2} < t < 10; \tfrac{1}{2} < s < 10; |x| + |y| < 10\}$$

Proof. Suppose that instead of $\|\psi\|_{-1}^2 \leqslant \text{const}$ we have $\|\psi\|_0^2 \leqslant \text{const}$. Let n be the function given by Corollary 3.7. Then for $t < t_1$ for some t_1 fixed

$$E^{\psi}[\|\phi - n\|_{-1}^2(t) > N^{-\delta/6}] \leqslant \exp(-\text{const}\cdot N^{\delta/8}) \tag{4.8}$$

By the Schwartz inequality

$$|[(\phi - n) * \omega_{1-\delta/12}](x, t) \leqslant \|(\phi - n)(t)\|_{-1} \|\omega_{1-\delta/12}(x + \cdot)\|_1$$
$$\leqslant \text{const} \cdot N^{\delta/8} \|\phi - n\|_{-1}(t) \tag{4.9}$$

Here we have used the fact that

$$\|\omega_{1-\delta/12}\|_1 \leqslant \text{const} \cdot [(N^{-\delta/12})^{-2} \cdot N^{\delta/12}]^{1/2} = \text{const} \cdot N^{\delta/8} \tag{4.10}$$

Hence

$$E^{\psi}\{|(\phi - n) * \omega_{1-\delta/12}(x, t)| > N^{-\delta/24}\} \leqslant \exp(-\text{const} \cdot N^{\delta/12}) \tag{4.11}$$

Similarly, the same bound holds if $\delta/12$ is replaced by $\delta/24$. Note that in the range of (x, t) we are interested, $n(x, t)$ is a smooth function (Corollary A.2). Lemma 4.4 follows by taking the intersections of events $\{|\omega_{1-\delta/12} * (\phi - n)(x, t)| < N^{-\delta/24}\}$ and $\{|\omega_{1-\delta/24} * (\phi - n)(y, x)| < N^{-\delta/24}\}$.

Finally we have to remove the condition $\|\psi\|_0^2 < \text{const}$ by $\|\psi\|_{-1}^2 < \text{const}$. But this follows from Lemma 4.3 and simple stropping time arguments. ∎

Our strategy to prove Theorem 4.1 is to use Lemma 4.4 inductively for all scales. For this purpose, we have to check condition $\|\psi\|_{-1} \leqslant \text{const}$ after rescaling. The following lemma will be used in relating norms in different scales.

Lemma 4.5. Define the scaled norm $\|u\|_{1,\beta}^2$ by

$$\|u\|_{-1,\beta}^2 = \iint u(\beta x)\, u(\beta y)\, K(x, y)\, dx\, dy$$
$$= \iint u(x)\, u(y)\, K(x/\beta, y/\beta)\, dx\, dy\, \beta^{-2} \tag{4.12}$$

Here $K = K_\alpha$ is defined in (2.26). Then for $\beta < 1$ one has the bound

$$\|u\|_{-1,\beta}^2 \leqslant 2\|u\|_{-1}^2 \beta^{-3} \tag{4.13}$$

Proof. By variational principle ($\alpha = 1$ in K_α for simplicity)

$$\|u\|_{-1,\beta}^2 = \sup_v \left\{\int u(\beta x)\, v(x) e^{-\theta(x)}\, dx - \int [(\partial v/\partial x)^2 + v^2(x)]\, dx\right\}$$

Changing variables $\beta x \to x$, $\beta^2 v(x/\beta) = w$,

$$\|u\|_{-1,\beta}^2 = \beta^{-3} \sup_w \left\{\int u(x)\, w(x) e^{\theta(x/\beta)}\, dx - \int [(\partial w/\partial x)^2 + \beta^{-2} w^2(x)]\, dz\right\}$$

Let $\chi = w \exp[-\theta(x/\beta) + \theta(x)] \leqslant w$. Then

$$\left(\frac{\partial \chi}{\partial x}\right)^2 + \chi^2 \leqslant 2\left(\frac{\partial w}{\partial x}\right)^2 + 2\beta^{-2} w^2 \leqslant 2\left[\left(\frac{\partial w}{\partial x}\right)^2 + \beta^{-2} w^2\right]$$

Hence

$$\|u\|^2_{-1,\beta} \leqslant \beta^{-3} \sup_\chi \int u(x)\, \chi(x) e^{-\theta(x)}\, dx - \frac{1}{2}\int \left[\left(\frac{\partial \chi}{\partial x}\right)^2 + \chi^2\right] dx$$

$$\leqslant 2\beta^{-3} \|u\|^2_{-1} \quad \blacksquare$$

Together with (4.9) we have the following result.

Corollary 4.6. Suppose that $\|\psi\|^2_{-1} \leqslant \text{const}$. Then there is a constant C_1 such that for $1/2 < t < 10$

$$E^{\psi}\{\|\phi(t)\|^2_{-1,N^{-\delta/24}} > C_1\} < \exp(-\text{const} \cdot N^{\delta/8}) \tag{4.14}$$

Here δ is chosen as in Lemma 4.4.

Proof. As in the proof of Lemma 4.4, let us first assume that $\|\psi\|^2_0 \leqslant \text{const}$. Then (4.9) holds. Hence by (4.13) we have (4.14) and this concludes Corollary 4.6. To replace the assumption $\|\psi\|^2_0 \leqslant \text{const}$ by $\|\psi\|^2_{-1} \leqslant \text{const}$ one can employ the stopping time argument as in Lemma 4.4. $\blacksquare$

Note that t has to stay away from zero because a stopping time argument as in Lemma 4.4 has to be used. Corollary 4.6 asserts that starting from the boundedness of $\|\cdot\|_{-1}$ at time $t = 0$, one actually obtains the boundedness of $\|\cdot\|_{-1}$ in smaller scale at later time. So we can repeat this argument to conclude $\|\cdot\|_{-1}$ is bounded for all scales. One technical point is that the improvement in the $\|\cdot\|_{-1}$ comes only after some period of time. But it is an easy consequence of Corollary 3.7 that the $\|\cdot\|_{-1}(t)$ will not be worse than $\|\cdot\|_{-1}(0)$ for $0 < t \leqslant 1$. More precisely, since $\|\phi\|^2_{-1}$ is bounded for time $0 < t < \exp(N^{\varepsilon/4})$ for some $\varepsilon > 0$ by Corollary 4.2, we have by Corollary 4.6 that $\|\phi\|^2_{-1,N^{-\sigma/24}}$ is bounded for all time t with $\frac{1}{2} < t < \exp(N^{\gamma/4})$. Now we can apply Corollary 4.6 again at the scale $N^{-\delta/24}$ and conclude that ϕ has to be bounded in the scale $N^{-\delta/24}(N^{1-\delta/24})^{-\delta/24}$. By repeating this procedure one concludes that ϕ is bounded for arbitrary small scale for time, say, $1 < t < \exp(N^{\tau/4})$. Without further assumption on initial data, we should not be able to conclude the boundedness of $\|\phi\|_{-1,N^{-\alpha}}$ for the initial layer $0 < t < 1$. However, if one makes assumption on initial data, it is easy to check by Corollary 3.7 that it does propagate for a finite time. Hence we have the following lemma.

Lemma 4.7. Suppose that the initial condition ψ satisfies that for some $\tau > 0$

$$\sup_x \|\tau_x \psi\|_{-1, N^{-\tau}} \leqslant C_1 \tag{4.15}$$

Then there is an $\varepsilon > 0$ so that

$$P^{\psi}\{\sup_{0 < t < \exp(N^{\gamma/4})} \sup_x \|\tau_x \phi(t)\|^2_{-1, N^{-\tau}} > C_2\} < \exp(-N^{\gamma}) \tag{4.16}$$

for some constant C_2.

Proof. Let us first prove (4.16) with $\sup_x$ and $\sup_t$ outside P^{ψ}. This indeed is just a corollary of Lemmas 4.1–4.6 as explained earlier.

We now move the sup t and sup x inside the probability. For sup x this is trivial because the dependence of ϕ on x is discrete with $x = i/N$ and the probability is exponentially small in N. Hence we are allowed to take N unions without changing the order of magnitude of the probability. To move the sup t inside is slightly harder, as t is continuous. Again because the probability is exponentially small, we are allowed to take unions of events with the number of events smaller than, say, $e^{N^{\gamma/2}}$. Therefore

$$P\{\sup_x \sup_{0 < j < \exp[N^{\gamma/2}], j \in \mathbb{N}} \|\tau_x \phi(j \exp[-\varepsilon/4])\|^2_{-1, N^{-\tau}} > C\} < \exp(-C_2 N^{\gamma})$$

To conclude (4.14), it remains to prove some very weak continuity of $\|\phi(t)\|^2_{-1, N^{-\tau}}$. By Lemma 4.2 and Ito's formula

$$P\{\sup_{\substack{|t_1 - t_2| < \exp(-N^{\gamma/4}) \\ 0 < t_i < \exp(N^{\gamma/4})}} | \|\tau_x \phi(t_1)\|_{-1, N^{-\tau}} - \|\tau_x \phi(t_2)\|_{-1, N^{-\tau}}| > \varepsilon\} < \exp(-C_2 N^{\gamma})$$

This provides the continuity needed to prove (4.16). ∎

Remark. This proof contains two arguments we will use several times in this paper. The first one is to remove the restriction on small t by arguing independently for small t with the help of assumptions on initial data. The other concerns moving sup of x and t inside. Because these arguments are all similar in all contexts in this paper, we shall not repeat them later.

Lemma 4.8. Suppose ϕ satisfies (2.1) with periodic boundary condition. Suppose the initial data ψ satisfies that $\|\psi\|_0^2 \leqslant$ const. Then for any positive constant $q > 0$ there is a $t_0 > 0$ such that for β small enough and some $\sigma > 0$ ($\bar{\psi} = N^{-1} \sum_{j=1}^N \psi_j$)

$$E^{\psi}[\sup_{t_0 < t < \exp(N^{\gamma/4})} \sup_x |(\omega_{1-\beta} * \phi)(x, t) - \bar{\psi}| > q] < \exp(-N^{\gamma}) \tag{4.17}$$

Proof. Recall first the bound (4.11). Note that for n satisfying (3.11) with periodic boundary condition there exists a $t_0 > 0$ such that

$$\sup_{t_0 < t < 100t_0} \sup_{x,y} |n(x, t) - n(y, t)| < \omega/4 \tag{4.18}$$

See Corollary A.2 for a proof of (4.18). Together with (4.11) we have proved (4.17) for $t_0 < t < 100t_0$. To extend (4.17) for all time, one can use Lemma 4.2, which ensures that the assumption $\|\psi\|_0^2 < \text{const}$ holds if we start from any $t < \exp(\text{const} \cdot N)$. This proves Lemma 4.8. ∎

Proof of Theorem 4.1. By Lemma 4.7 we have that the assumption of Lemma 4.4, namely $\|\psi\|_{-1}^2 \leqslant \text{const}$, holds for all scales bigger than $N^{\varepsilon-1}$. Hence its conclusion holds up to that scale. This implies that $|\omega_{1-\delta} * \phi(x, t) - (\omega_{1-\varepsilon} * \phi)(x, y)| < N^{-z}$ for some $z > 0$ with probability $1 - \exp(-N^{\gamma})$ for some $\gamma > 0$. (Here δ is chosen as in Lemma 4.4.) In other words, we have related the local density for scale $N^{\varepsilon-1}$ to scale $N^{-\delta}$ with δ a small positive constant. So in order to prove (4.3), it suffices to prove that for any $q > 0$

$$P\{\sup_{0 < t < \exp(N^{\gamma/4})} \sup_{x} |(\omega_{1-\delta} * \phi)(x, t) - u(x, t)| > q\} < \exp(-N^{\gamma}) \tag{4.19}$$

Clearly, (4.19) holds for $t > t_0$ by (4.7) and (A.9). For $0 \leqslant t \leqslant t_0$, (4.19) is just a corollary of Corollary 3.7 and the Chebyshev inequality. This concludes Theorem 4.1. ∎

5. PROOF OF THEOREM 1

We have proved Theorem 4.1 in Section 4 by using large-deviation bounds, especially Corollary 3.7 from Section 3. Theorem 1 is the corresponding version of Theorem 4.1 in the long-range models (taking into account the remark after Theorem 1 which replaces the Hamiltonian H by $\hat{H}$). Though its assumptions (ii) and (iii) appear different from assumptions A and B of Theorem 4.1, they are equivalent. Hence it is natural to try the same approach. Indeed, we can follow the same proof without too many modifications. Let us first extend results in Section 3 to long-range models.

First of all Lemma 3.2 holds if we make the following changes. Equation (3.11) has to be replaced by

$$\partial_t n(x, t) = [h'(n(x, t)) + g'(n(x, t)) - (J * n)(x, t)]_{xx} \tag{5.1}$$

where g is the function in (2.22). In the definitions of g_1 and g_2 (3.15) and (3.16), the Hamiltonian H has to be replaced by H_0 of (2.22). Finally, we choose $\delta = \tau$ with N^{τ} denoting the range of $\bar{\phi}_{\sigma}$ in (2.22). As the boxes in the definition of H' are disjoint, the integration in the definition of G_i [before (3.15)] has to be interpreted as a sum over these disjoint boxes rather than integrating over overlapping boxes. Similar interpretations have to be imposed also in the proof of Lemma 3.2. With these interpretations and modifications, Lemma 3.2 holds with the same proof.

Next Lemmas 3.3–3.6 hold without change.

Note that in (3.66) we have to use a spectral gap for product measure with constraint $\sum \phi_i = \text{const}$. For long-range models, Corollary 6.3 provides a bound on the Dirichlet form relative to $\exp(-H_0 - H')$ which is a product measure with constraint $\sum \phi_i = \text{const}$ when restricted to a box of size N^τ. So we return to the same problem as in the product case with constraint $\sum \phi_i = \text{const}$. Indeed, the purpose of Lemma 6.2 is to avoid the discussion involving the spectral gap for long-range models, which, even though it is correct, requires a proof. The long-range nature becomes significant only in the bound (3.56), which affects (3.62). By Corollary 6.3, q is bounded by N^{2-2a} [a defined in (2.3)]. Hence e in (3.62) is equal to $2a$ and the condition $\delta \leqslant e/5$ is satisfied by our choice

$$\delta = \tau = a/20 \tag{5.2}$$

Therefore, Corollaries 3.7 and 3.8 hold.

Finally we sketch modifications needed for the argument in Section 4. First of all, all results concerning differential Equation (3.11) have to be proved for (5.1). This will be done in the Appendix. Besides this, almost all results in Section 4 depends only on Corollaries 3.7 and 3.8 and can be proved in the same way. One can check that the large-field cutoffs from Lemmas 4.2 and 4.3 hold for a very general class of models and, in particular, the long-range model we consider. Lemma 4.4 is just a corollary of (3.72) and the Chebyshev inequality. Lemma 4.5 concerns a general fact of the H_1 norm and is independent of dynamics. The next step in Section 4 is the rescaling and repeating the same argument. It should be emphasized that the range of interactions N^a increases after each rescaling. Hence the constant q in (3.56) decreases because of the bound (6.31). Therefore one has a better estimate after each rescaling. This shows that the long-range interaction H_L does not affect the proofs in Section 4.

One may worry about the effect of the Hamiltonian H' of (2.22). But the role played by H' is indeed minimal and is used only to keep the local average of field $\omega_\delta * \phi$ away from the unstable region $\omega_\delta * \phi \geqslant \underline{m} - \varepsilon_1/2$ [(2.17), (2.24)] for δ of the order a with N^a the range of the interaction. Once we have proved (2.21) for scale up to N^a, we can simply drop the term involving $\partial H'/\partial\phi$ because the combined contribution of $\partial H'/\partial\phi$ in the terms Ω_1 and Ω_2 [(3.20), (3.21)] is of definte sign (by the convexity of g)

and the local average of field $\omega_q * \phi$ is now in the region $\underline{m} - \varepsilon_1/2$. For scale bigger than the range of interaction N^a, one uses the choice (5.2). The proof has just been explained in the previous paragraph. This concludes the proof of Theorem 2.1.

6. ENTROPY PRODUCTION IN $\mathbb{Z}^d$

As explained in Section 4, we have to rescale the system (2.1) many times. After such rescalings, we are practically in infinite volume. Since we base our method on the Dirichlet form and entropy, we need a bound on entropy productions for the infinite systems. This has been done by Fritz[3] for zero-range models. We shall reprove Fritz's result with a slightly better bound for short-range models. It will then be extended to long-range models. In order to focus the discussion on central issues, we shall consider only the lattice $[1, M] \subset \mathbb{Z}^1$ with all estimates uniformly in M. Certainly all results in this section can be generalized to $\mathbb{Z}^d$.

Let μ be the Gibbs state with the Hamiltonian

$$H = \sum_{i=1}^{M} V(\phi_i) + F(\phi_i, \phi_{i+1}) \tag{6.1}$$

with F being a bounded function. Suppose that f_t solves the forward equation

$$\partial_t f_t = L f_t \tag{6.2}$$

with $L = L^*$ given by the Dirichlet form

$$-\int fLf\, d\mu = N^2 \sum_{i=1}^{\infty} \int [(\partial_i - \partial_{i+1}) f]^2 f^{-1}\, d\mu \tag{6.3}$$

Denote by $\mathscr{F}_j$ the σ-field generated by $\phi_1, \ldots, \phi_j$ and by f_j the conditional expectation with respect to $\mathscr{F}_j$, namely

$$f_j = E^\mu[f \mid \mathscr{F}_j] \tag{6.4}$$

For the potential V we shall assume throughout this section that

$$\lim_{|\phi| \to \infty} [|V'(\phi)|^2 + |\phi|^2]/V(\phi) < \infty \tag{6.5}$$

One immediately checks that for any $i > 0$

$$\delta^{-1} \log E^\mu[\exp\{\delta(\partial H/\partial\phi_i - a)\}] \leqslant \text{const} \cdot \delta \tag{6.6}$$

where $a = E^{\mu}[\partial H/\partial\phi_i]$. The bound (6.6) is interesting only when δ is small. To prove it let $l = \delta^{-s}$ and decompose the expection into $|\phi_i| \leqslant l$ and $|\phi_i| > l$. In the region $|\phi_i| > l$, the expectation in (6.6) is bounded by e^{-cl}, which is negligible when $\delta \to 0$. For the region $|\phi_i| \leqslant l$ one can expand the exponential and checks that (6.6) holds.

An important corollary of (6.6) is that for any density f relative to μ one has the following bound by the entropy inequality:

$$E^{\mu}[f; \partial H/\partial\phi_i] \leqslant \delta^{-1} \log E^{\mu}[\exp\{\delta(\partial H/\partial\phi_i - a)\}] + \delta^{-1} S(f/\mu) \quad (6.7)$$

Here the entropy $S(f|\mu)$ is defined by

$$S(f/\mu) = \int f \log f \, d\mu \quad (6.8)$$

Optimizing δ, one has that

$$E^{\mu}[f; \partial H/\partial\phi_i]^2 \leqslant \text{const} \cdot S(f/\mu) \quad (6.9)$$

This bound holds in very general situations as long as (6.6) holds. Let S_j be the entropy of f_j relative to μ, i.e.,

$$S_j = S(f_j) = E^{\mu}[f \log f_j] \quad (6.10)$$

Define the specific entropy by

$$s(f) = N^{-2} \sum_{j=1}^{\infty} e^{-\theta(j/N)} S(f_j) \quad (6.11)$$

with θ given by (2.26).

Theorem 6.1.[3] Suppose f_t satisfies the forward Equation (6.2). Then the entropy production of f_t satisfies

$$\frac{ds(f_t)}{dt} \leqslant -\sum_j \left[\sum_{i=1}^{j-1} E^{\mu}\{[(\partial_i - \partial_{i+1}) f_j]^2 f_j^{-1}\} \right] e^{-\theta(j/N)} + \text{const} \cdot s(f_t) \quad (6.12)$$

Proof. By definition (6.10) the entropy production is

$$\frac{dS_j}{dt} = \int Lf \log f_i \, d\mu + \int f \left(\frac{d}{dt} f_j\right) f_j^{-1} \, d\mu \quad (6.13)$$

Clearly, the last term is zero. So we can write the entropy production as

$$\begin{aligned} \frac{dS_j}{dt} &= -N^2 \sum_{i=1}^{j} \int [(\partial_i - \partial_{i+1}) f][(\partial_i - \partial_{i+1}) f_j] f_j^{-1} \, d\mu \\ &= -N^2 \sum_{i=1}^{n} E^{\mu}\{(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j\}\{(\partial_i - \partial_{i+1}) f_j\} f_j^{-1}] \end{aligned} \quad (6.14)$$

By definition of f_j,

$$(\partial_i - \partial_{i+1}) f_j = E[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j] - E[f; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j] \quad (6.15)$$

where

$$E[a; b \mid \mathscr{F}_j] \equiv E[ab \mid \mathscr{F}_j] - E[a \mid \mathscr{F}_j]\, E[b \mid \mathscr{F}_j] \quad (6.16)$$

In particular, since H is nearest-neighbored,

$$(\partial_i - \partial_{i+1}) f_j = E[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j], \qquad i \leqslant j-2 \quad (6.17)$$

Therefore, we can rewrite (6.14) as

$$\frac{dS_j}{dt} = -N^2 \sum_{i=1}^{j-2} \int [(\partial_i - \partial_{i+1}) f_j]^2 f_j^{-1}\, d\mu + \Omega_j \quad (6.18)$$

where Ω_j is defined by

$$\Omega_j = -N^2 \sum_{i=j-1}^{j} E^\mu\{E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j][(\partial_i - \partial_{i+1}) f_j] f_j^{-1}\} \quad (6.19)$$

By (6.15), we can decompose Ω_j as

$$\begin{aligned} \Omega_j = & -N^2 \sum_{i=j-1}^{j} E^\mu\{E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j]^2 f_j^{-1}\} \\ & + N^2 \sum_{i=j-1}^{j} E^\mu\{E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j]\, E[f; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j] f_j^{-1}\} \end{aligned} \quad (6.20)$$

The first term on the right side of (6.20) is negative. The second term can be bounded by the Schwartz inequality as

$$\begin{aligned} \Omega_j \leqslant & \, \varepsilon N^3 \sum_{i=j-1}^{j} E^\mu\{E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j]^2 f_j^{-1}\} \\ & + \varepsilon^{-1} N \sum_{i=j-1}^{j} E^\mu\{E^\mu[f; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j]^2 f_j^{-1}\} \end{aligned} \quad (6.21)$$

for all $\varepsilon > 0$. Since for $i \leqslant j < k-3$,

$$\begin{aligned} E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_j]^2 &= E^\mu[E^\mu[(\partial_i - \partial_{i+1}) f \mid \mathscr{F}_k] \mid \mathscr{F}_j]^2 \\ &= E^\mu[(\partial_i - \partial_{i+1}) f_k \mid \mathscr{F}_j]^2 \\ &\leqslant E^\mu[\{(\partial_i - \partial_{i+1}) f_k\}^2 f_k^{-1} \mid \mathscr{F}_j] f_j \end{aligned} \quad (6.22)$$

the first term on the right side of (6.21) is bounded by

$$2\varepsilon N^2 \sum_{i=j-1}^{j} \sum_{k=j+3}^{j+N} E^{\mu}[\{(\partial_i - \partial_{i+1}) f_k\}^2 f_k^{-1}] \tag{6.23}$$

The second term is slightly harder. We shall only bound the $i=j$ term. The other term is similar. By definition $(i=j)$,

$$E^{\mu}[f; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j] = E^{\mu}[f_{i+3}; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j] \tag{6.24}$$

Let $u = f_{j+3}/f_j$. Then u is a probability density relative to $E^{\mu}[\cdot \mid \mathscr{F}_j]$. By (6.9)

$$E^{\mu}[f_{j+3}; (\partial_j - \partial_{j+1}) H \mid \mathscr{F}_j]^2 f_j^{-2} \leqslant \text{const} \cdot E^{\mu}[f_{j+3} \log(f_{j+3}/f_j) \mid \mathscr{F}_j] f_j^{-1} \tag{6.25}$$

Hence the last term in (6.20) is bounded by

$$\begin{aligned} &\text{const} \cdot \varepsilon^{-1} N^2 E^{\mu}\{E^{\mu}[f_{j+3} \log(f_{j+3}/f_j) \mid \mathscr{F}_j]\} \\ &\quad = \text{const} \cdot \varepsilon^{-1} N^2 (S_{j+3} - S_j) \end{aligned} \tag{6.26}$$

To summarize, we have proved that

$$\begin{aligned} \frac{dS_j}{dt} \leqslant &-N^2 \sum_{i=1}^{j-2} E^{\mu}\{[(\partial_i - \partial_{i+1}) f_j]^2 f_j^{-1}\} \\ &+ 2\varepsilon N^2 \sum_{i=j-1}^{j} \sum_{k=j+3}^{j+N} E^{\mu}\{[(\partial_i - \partial_{i+1}) f_k]^2 f_k^{-1}\} \\ &+ \text{const} \cdot \varepsilon^{-1} N (S_{j+3} - S_j) \end{aligned} \tag{6.27}$$

Theorem 6.1 follows by multiplying (6.27) by $N^{-2} \exp[-\theta(j/N)]$ and summing over j and choosing ε small enough. Note that the factor N in the last term of (6.27) disappears after the summation by parts in j. ∎

Remark. The previous proof works also for $d \geqslant 1$ with the following modifications. In (6.19), the summation over i becomes over the boundary of cubes of size i. The first term on the right side of (6.20) can be dealt with in the same way. For the second term, we replace the right side of (6.24) by $E^{\mu}[\tilde{f}_i; (\partial_i - \partial_{i+1}) H \mid \mathscr{F}_j]$ with $\tilde{f}_i = E^{\mu}[f \mid \tilde{\mathscr{F}}_i]$. Here $\tilde{\mathscr{F}}_i$ denotes the σ-field generated by $\mathscr{F}_i$ and ϕ_k with $|k-i| \leqslant 2$. One can then follow the rest of the proof to conclude Theorem 6.1 for $d \geqslant 1$.

We now return to the case that μ is given by the Hamiltonian $\hat{H}$ in (2.22). An entropy production bound similar to Theorem 6.1 still holds with a similar proof. Our goal is, however, to bound the entropy production relative to the short-range equilibrium state. To be more specific, let dv be the Gibbs state with the Hamiltonian H_s and denote the Radon–Nikodym derivative by

$$h = dv/d\mu = \exp(H_L)/\text{normalization} \tag{6.28}$$

Let $g = f/h$ and let

$$g_j = E^v[g \mid \mathscr{F}_j] \tag{6.29}$$

Define the entropy $S(g_j) = S(g_j/v)$ and the specific entropy

$$s(g/v) = N^{-2} \sum_{j=1}^{\infty} e^{-2\theta(j/N)} S_j \tag{6.30}$$

Lemma 6.2. Suppose that f_t satisfies the forward equation (6.2) with H replaced by $\hat{H}$. Then with the previous notation the entropy production is bounded by $[g = g_t = g(t)]$

$$\begin{aligned} \frac{ds(t)}{dt} &= \frac{ds(g(t)/v)}{dt} \\ &\leqslant -\frac{1}{4} \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} e^{-2\theta(j/N)} \int [(\partial_i - \partial_{i+1}) g_j]^2 g_j^{-1}\, dv \\ &\quad + \text{const} \cdot s(t) + \text{const} \cdot N^{2-2a} \int \|\phi\|_0^2 g\, dv \end{aligned} \tag{6.31}$$

Here a is the range of interaction in (2.22) and $\|\cdot\|_0$ denotes the norm in (2.27).

Corollary 6.3. Suppose, in addition, that

$$E^f\left\{\int_0^t \|\phi\|_0^2(s)\, ds\right\} \leqslant \text{const}$$

Then

$$\begin{aligned} s(t) + \frac{1}{4} \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} e^{-2\theta(j/N)} \int_0^t ds \int [(\partial_i - \partial_{i+1}) g_j(s)]^2 g_j(s)^{-1}\, dv \\ \leqslant [\text{const} \cdot \exp(\text{const} \cdot t)]\,[s(0) + N^{2-2a}] \end{aligned} \tag{6.32}$$

Remark. The Hamiltonian $\hat{H}$ really has two long-range parts. In Lemma 6.2 we assume that H' of (2.20) vanishes. Lemma 6.2 also holds with the same proof if we define dv in (6.28) as the Gibbs state with Hamiltonian $H_s + H'$.

Proof. Step 1. Recall that g_t satisfies the equation[18]

$$\frac{\partial g_t}{\partial t} = L_\nu^* g_t \tag{6.33}$$

where L_ν^* is defined by the identity

$$\int \xi(L_\nu^* \eta)\, d\nu = \int (L\xi)\eta\, d\nu \tag{6.34}$$

Hence we can compute the entropy produce ($g_j(t) = (g_t)_j = E^\nu[g_t \mid \mathscr{F}_j]$)

$$\frac{d}{dt} S\left(\frac{g_j(t)}{\nu}\right) = \int L_\nu^* g(t) \log g_j(t)\, d\nu + \int g(t) \left(\frac{dg_j(t)}{dt}\right) g_j^{-1}\, d\nu \tag{6.35}$$

The second term on the right side vanishes as in Theorem 6.1. For the first term we integrate L_ν^* by parts to have

$$\int L_\nu^* g(t) \log g_j(t)\, d\nu = \int g(t)[L \log g_j(t)]\, d\nu$$

$$= \int g(t)[L \log g_j(t)] h\, d\mu \tag{6.36}$$

From now on, we shall omit the t variable. By definition of L, the last term is equal to

$$-N^2 \sum_i \int [(\partial_i - \partial_{i+1})(gh)][(\partial_i - \partial_{i+1}) g_j]\, g_j^{-1}\, d\mu = \Omega_1 + \Omega_2 \tag{6.37}$$

where Ω_1 and Ω_2 are given by

$$\Omega_1 = -N^2 \sum_{i=1}^{j} \int [(\partial_i - \partial_{i+1}) g][(\partial_i - \partial_{i+1}) g_j]\, g_j^{-1}\, d\nu \tag{6.38}$$

$$\Omega_2 = -N^2 \sum_{i=1}^{j} \int g[(\partial_i - \partial_{i+1}) \log h][(\partial_i - \partial_{i+1}) g_j]\, g_j^{-1}\, d\nu \tag{6.39}$$

Note that the long-range contribution appeared only in $\log h$. Also, ν is a product measure. We can thus bound Ω_1 as in Theorem 6.1. Our remaining task is to bound Ω_2.

Step 2. We now bound Ω_2. First we decompose $\Omega_2 = \Omega_3 + \Omega_4$ with Ω_4 denoting the $i = j$ term in (6.39) and Ω_3 denoting the rest, namely,

$$\Omega_3 = -N^2 \sum_{i=1}^{j-1} \int dv\, E^\nu[\{(\partial_i - \partial_{i+1}) \log h\}\, g \mid \mathscr{F}_j][(\partial_i - \partial_{i+1})\, g_j]\, g_j^{-1} \tag{6.40}$$

$$\Omega_4 = -N^2 \int dv\, E^\nu[\{(\partial_j - \partial_{j+1}) \log h\}\, g \mid \mathscr{F}_j][(\partial_j - \partial_{j+1})\, g_j]\, g_j^{-1} \tag{6.41}$$

By the Schwartz inequality we can bound Ω_3 by

$$\Omega_3 \leqslant \delta N^2 \sum_{i=1}^{j-1} \int dv\, [(\partial_i - \partial_{i+1})\, g_j]^2\, g_j^{-1}\, dv + \Omega_5 \tag{6.42}$$

where Ω_5 is defined by

$$\Omega_5 = \delta^{-1} N^2 \sum_{i=1}^{j-1} \int dv\, E^\nu[\{(\partial_i - \partial_{i+1}) \log h\}\, g \mid \mathscr{F}_j]^2\, g_j^{-1} \tag{6.43}$$

Again by the Schwartz inequality Ω_5 is bounded by

$$\begin{aligned} \Omega_5 &\leqslant \delta^{-1} N^2 \sum_{i=1}^{j-1} \int dv\, E^\nu[\{(\partial_i - \partial_{i+1}) \log h\}^2 g \mid \mathscr{F}_j]\, E^\nu[g \mid \mathscr{F}_j]\, g_j^{-1} \\ &= \delta^{-1} N^2 \sum_{i=1}^{j-1} \int \{(\partial_i - \partial_{i+1}) \log h\}^2 g\, dv \end{aligned} \tag{6.44}$$

We now bound the boundary term Ω_4. By the identity (6.15), we can replace $(\partial_i - \partial_{i+1})\, g_j$ by

$$E^\nu[(\partial_i - \partial_{i+1})\, g_k \mid \mathscr{F}_j] - E^\nu[V'(\phi_{j+1}); g_{j+2} \mid \mathscr{F}_j]$$

for any $k \geqslant j+2$. Hence we can write $\Omega_4 = \Omega_6 + \Omega_7$ with

$$\Omega_6 = N \sum_{k=j+2}^{j+N+2} \int E^\nu[\{(\partial_j - \partial_{j+1}) \log h\}\, g \mid \mathscr{F}_j][(\partial_j - \partial_{j+1})\, g_k]\, g_j^{-1}\, dv \tag{6.45}$$

$$\Omega_7 = -N^2 \int E^\nu[\{(\partial_j - \partial_{j+1}) \log h\}\, g \mid \mathscr{F}_j]\, E^\nu[V'(\phi_{j+1}); g \mid \mathscr{F}_j]\, g_j^{-1}\, dv \tag{6.46}$$

By the Schwartz inequality Ω_6 is bounded by [cf. (6.44)]

$$\Omega_6 \leqslant \delta \sum_{k=j+2}^{j+2+N} \left[\int \{(\partial_j - \partial_{j+1}) \log h\}^2 g\, dv + N^2 \int \{(\partial_j - \partial_{j+1})\, g_k\}^2 g_k^{-1}\, dv \right] \tag{6.47}$$

Similarly, Ω_7 is bounded by

$$\Omega_7 \leqslant \varepsilon^{-3} \int \{(\partial_j - \partial_{j+1}) \log h\}^2 g \, d\nu + \Omega_8 \tag{6.48}$$

with Ω_8 given by

$$\begin{aligned} \Omega_8 &= N \int E^{\nu}[V'(\phi_{j+1}); g \mid \mathscr{F}_j]^2 g_j^{-1} \, d\nu \\ &\leqslant \text{const} \cdot N[S(g_{j+1} \mid \nu) - S(g_j \mid \nu)] \end{aligned} \tag{6.49}$$

Here we have used (6.9).

To summarize, we have bounded Ω_2 by

$$\begin{aligned} \Omega_2 \leqslant{}& \delta N^2 \sum_{i=1}^{j-1} \int [(\partial_i - \partial_{i+1}) g_j]^2 g_j^{-1} \, d\nu \\ &+ \delta N \sum_{k=j+2}^{j+2+N} \int [(\partial_j - \partial_{j+1}) g_k]^2 g_k^{-1} \, d\nu \\ &+ N^2 \sum_{i=1}^{j-1} \int [(\partial_i - \partial_{i+1}) \log h]^2 g \, d\nu \\ &+ N^3 \int [(\partial_i - \partial_{i+1}) \log h]^2 g \, d\nu \\ &+ \text{const} \cdot N[S(g_{j+1}/\nu) - S(g_j/\nu)] \end{aligned} \tag{6.50}$$

Step 3. Combining steps 1 and 2, we have a bound on the entropy production dS_j/dt. As in the proof of Theorem 6.1, we multiply this bound by $N^{-2} \exp[-\theta(j/N)]$ and then sum over j. By choosing δ small, we arrive at

$$\begin{aligned} \frac{ds(t)}{dt} \leqslant{}& -\frac{1}{4} \sum_j \sum_{i=1}^{j-1} e^{-\theta(j/N)} \int [(\partial_i - \partial_{i+1}) g_j]^2 g_j^{-1} \, d\nu \\ &+ \text{const} \cdot N \sum_{j=1}^{\infty} e^{-\theta(j/N)} \int [(\partial_j - \partial_{j+1}) \log h]^2 g \, d\nu \\ &+ \text{const} \cdot s(t) \end{aligned} \tag{6.51}$$

Note that we sum over i up to $j-1$ in the first term of (6.51) as compared with summing up to $j-2$ in (6.27). This is because ν is a product measure here but has range 1 in (6.27). By the definition of h we have

$$N \sum_{j=1}^{\infty} e^{-\theta(j/N)} [(\partial_j - \partial_{j+1}) \log h]^2 \leqslant \text{const} \cdot N^{2-2a} \|\phi\|_0^2 \tag{6.52}$$

where a is the range of interaction defined in (2.3) and $\|\cdot\|_0$ is defined in (2.27). Therefore, the entropy production is bounded by $(g = g_t)$

$$\frac{ds(t)}{dt} \leqslant -\frac{1}{4} \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} e^{-\theta(j/N)} \int [(\partial_i - \partial_{i+1}) g_j]^2 g_j^{-1} \, dv + \text{const} \cdot s(t) + \text{const} \cdot N^{2-2a} \int \|\phi\|_0^2 g \, dv \tag{6.53}$$

This proves Lemma 6.2. ∎

APPENDIX

In this Appendix we collect some results for differential equation (3.2). Since they are well known if the $J * n$ term is dropped, we shall only give a sketch of the proofs.

Lemma A.1. Let n be a solution to the equation in $\mathbb{R}$

$$\begin{aligned} n_t &= (a(n) - J * n)_{xx} \\ n(0, x) &= n_0 \end{aligned} \tag{A.1}$$

where a and J satisfy for some $\varepsilon > 0$

$$\varepsilon^{-1} > a' > 1 + \varepsilon = \int J(x) \, dx \tag{A.2}$$

Then we have the following bounds provided that α in (2.26) is chosen small enough:

$$\|n\|_0^2(T) + \frac{\varepsilon}{2} \int_0^T \|n_x\|_0^2(t) \, dt \leqslant C \exp(CT) \|n_0\|_0^2 \tag{A.3}$$

$$\|n_x\|_0^2(T) + \frac{\varepsilon}{2} \int_0^T \|n_t\|_0^2 \, dt \leqslant CT^{-1} \exp(CT) \|n_0\|_0^2 \tag{A.4}$$

Here the constant C depends on ε, α, and θ [defined in (2.26)].

Proof. Multiply (A.1) by $ne^{-2\theta(\alpha x)}$ and integrate over the space variable

$$\frac{1}{2} \frac{d}{dt} \int n^2 \, dx = \int n(a(n) - J * n)_{xx} \exp[-2\theta(\alpha x)] \, dx \tag{A.5}$$

Integrate the right side by parts,

$$\frac{1}{2}\frac{d}{dt}\int n^2 e^{-2\theta(\alpha x)}\,dx = -\int n_x[a'(n)n_x - J * n_x]\,e^{-2\theta(\alpha x)}\,dx$$

$$+2\alpha\int n[a'(n)n_x - J * n_x]\,\theta'(x)\,e^{-2\alpha\theta(x)}\,dx \qquad (A.6)$$

By assumption (A.2), the first term on the right side is bounded above by

$$-\varepsilon\int n_x^2 \exp[-2\theta(\alpha x)]\,dx$$

By Schwartz's inequality, the second term is bounded by

$$\alpha\,\text{const}\cdot(\|n_x\|_0^2 + \|n\|_0^2)$$

Choose α small enough. Using these bounds and Gronwall's inequality in (A.6), we have

$$\|n(T)\|_0^2 + \frac{\varepsilon}{2}\int_0^T \|n_x\|_0^2\,dt \leqslant \exp(\text{const}\cdot T)\,\|n_0\|_0^2$$

Here the const depends on α, θ, and ε. This concludes (A.3).

To prove (A.4), let us multiply (A.1) by $(a(n) - J * n)_t \exp[-2\theta(\alpha x)]$ and integrate the space variable,

$$\int n_t(a(n) - J * n)_t \exp[-2\theta(\alpha x)]\,dx$$

$$= -\frac{d}{dt}\int \frac{1}{2}(a(n) - J * n)_x^2 \exp[-2\theta(\alpha x)]\,dx$$

$$+2\alpha\int (a(n) - J * n)_t (a(n) - J * n)_x\,\theta'(x)\exp[-2\theta(\alpha x)]\,dx \qquad (A.7)$$

Here we have integrated the x by parts. The left side of (A.7) is bounded below by

$$\varepsilon\int n_t^2 \exp[-2\theta(\alpha x)]\,dx$$

The second term on the right side of (A.7) is bounded above by

$$\alpha\,\text{const}\cdot[\|n_t\|_0^2 + \|(a(n) - J * n)_x\|_0^2]$$

Using these bounds and choosing α small enough, we have by the Gronwall inequality

$$\|(a(n)-J*n)_x\|_0^2(T)+\frac{\varepsilon}{2}\int_s^T \|n_t\|_0^2\,dt$$
$$\leqslant \exp[\text{const}\cdot(T-s)]\|(a(n)-J*n)_x\|_0^2(s) \tag{A.8}$$

By (A.2), $\|a(n)-J*n)_x\|_0^2$ is equivalent to $\|n_x\|_0^2$. Finally we can integrate s from zero to $T/2$ and use (A.3) to have (A.4). ∎

Corollary A.2. Suppose n is a solution of (A.1) with the initial data n_0 satisfying $\|n_0\|_0^2<\text{const}$. Suppose that the space dimension $d=1$. Then at any time $t>0$, n_t is a smooth function.

Proof. From (A.4) and the Sobolev inequality we have that $n(t)$ is Hölder continuous. We can now follow the usual approach for parabolic equations to conclude the smoothness of n. Strictly speaking, this argument is not rigorous, because Lemma A.1 is only an *a priori* estimate. It is not hard, however, to provide a rigorous proof based on these *a priori* estimates (A.3), (A.4) with usual arguments for parabolic equations. ∎

Corollary A.3. Let n be a solution to (A.1) with periodic boundary condition and $\|n\|_0\leqslant\text{const}$. Then for any $\delta>0$ there is a $t_0>0$ such that

$$\sup_{t\geqslant t_0}\sup_x |n(x,t)-\bar{n}|\leqslant\delta \tag{A.9}$$

Proof. We follow the procedure for proving (A.3), but without the extra factor $\exp[-2\theta(\alpha x)]$, since we are now in a finite volume. Hence we have the bound

$$\frac{1}{2}\frac{d}{dt}\|n\|_0^2\leqslant -\varepsilon\|n_x\|_0^2 \tag{A.10}$$

Similarly, by arguing as in (A.7), we have

$$\|n_t\|_0^2\leqslant -\frac{1}{2}\frac{d}{dt}\|(a(n)-J*n)_x\|_0^2$$

Integrating it from s to T, we have

$$\tfrac{1}{2}\|(a(n)-J*n)_x\|_0^2(T)+\int_s^T \|n_t\|^2(t)\,dt\leqslant\tfrac{1}{2}\|(a(n)-J*n)_x\|_0^2(s)$$

Integrating s from 0 to $T/2$ and using (A.10) (its integration form), we have

$$\frac{1}{2}\|(a(n)-J*n)_x\|_0^2(T)\leqslant T^{-1}\int_0^{T/2}\|(a(n)-J*n)_x\|_0^2(s)\,ds\leqslant \varepsilon^{-1}T^{-1}\|n\|_0^2$$

Since $\|(a(n)-J*n)_x\|_0^2\geqslant \text{const}\cdot\|n_x\|_0^2$, we have proved that

$$\|n_x\|_0^2(T)\leqslant \text{const}\cdot T^{-1}\|n\|_0^2 \tag{A.11}$$

Clearly, Corollary A.3 follows from (A.11).

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